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Technical Notes

Computation of Eigensolution Derivatives for Nonviscously Damped Systems Using the Algebraic Method

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I. Introduction

THE computation of the eigensolution derivatives plays a significant role in dynamic model updating, structural design optimization, structural dynamic modification, damage detection and many other applications. The methods to calculate eigensolution derivatives are well established for undamped and viscous damped systems. These common methods can be divided into the modal method, Nelson's method, and the algebraic method.

Fox and Kapoor [1] first proposed the modal method for symmetric undamped systems by approximating the derivative of each eigenvector as a linear combination of all undamped eigenvectors. Later, Adhikari and Friswell [2] and Adhikari [3] extended the modal method to the more general asymmetric systems with viscous and nonviscous damping, respectively. To simplify the computation of eigensolution derivatives, Nelson [4] proposed a method, which requires only the eigenvector of interest by expressing the derivative of each eigenvector as a particular solution and a homogeneous solution for symmetric undamped systems. Later, Friswell and Adhikari [5] extended Nelson's method to symmetric and asymmetric systems with viscous damping. Recently, Adhikari and Friswell [6] extended Nelson's method to symmetric and asymmetric nonviscously damped systems. However, Nelson's method is lengthy and clumsy for programming. Lee et al. [7] derived an efficient algebraic method, which has a compact form to compute the eigensolution derivatives by solving a nonsingular linear system of algebraic equations for symmetric systems with viscous damping. Later, Guedria et al. [8] extended the algebraic method to general asymmetric systems with viscous damping. Recently, Chouchane et al. [9] wrote an excellent review of the algebraic method for symmetric and asymmetric systems with viscous damping and extended their method to the second-order and high-order derivatives of eigensolutions. In this note, the algebraic method will be extended to symmetric and asymmetric systems with nonviscous damping.

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The equations of motion describing free vibration of an N-degree-of-freedom (DOF) linear system with nonviscous (viscoelastic) damping can be expressed by [3,6]:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_{-\infty}^{t} \mathbf{g}(t-\tau)\dot{\mathbf{u}}(\tau)\,\mathrm{d}\tau + \mathbf{K}\mathbf{u}(t) = \mathbf{0} \tag{1}$$

Here $\mathbf{M}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are, respectively, the mass and stiffness matrices, $\mathbf{g}(t) \in \mathbb{R}^{N\times N}$ is the matrix of kernel functions, $t \in \mathbb{R}^{+}$ denotes time. In the special case, when $\mathbf{g}(t-\tau) = \mathbf{C}\delta(t-\tau)$ where $\mathbf{C}$ is a constant matrix, Eq. (1) reduces to the case of a viscously damped system. Therefore, the nonviscous damping model is considered as a further generalization of the familiar viscous damping. Taking the Laplace transform of the above equation gives

$$(s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K})\bar{\mathbf{u}}(s) = \mathbf{0} \quad \text{or} \quad \mathbf{D}(s)\bar{\mathbf{u}}(s) = \mathbf{0} \tag{2}$$

where $\bar{\mathbf{u}}(s) = L[\mathbf{u}(t)] \in \mathbb{C}^N$, $\mathbf{G}(s) = L[\mathbf{g}(t)] \in \mathbb{C}^{N\times N}$ and $L[]$ denotes the Laplace transform. The eigenvalues λ_i are roots of the characteristic equation

$$\det[s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K}] = \mathbf{0} \tag{3}$$

In general, the component of $\mathbf{G}(s)$ can be represented by the following form for the linear viscoelastic case [10]:

$$\mathbf{G}(s) = \frac{p_{jk}(s)}{q_{jk}(s)} \tag{4}$$

where $p_{jk}(s)$ and $q_{jk}(s)$ are finite-order polynomials in s and the degree of $q_{jk}(s)$ is not less than that of $p_{jk}(s)$. Under such assumptions, the order of the characteristic equation m is usually more than $2N$. Therefore, although the system has N DOF, the number of eigenvalues is more than $2N$. This is a major difference between viscously and nonviscously damped systems. According to Adhikari [11], the eigenvectors can be divided into the elastic modes (corresponding to N complex conjugate pairs of eigenvalues) and the nonviscous modes (corresponding to the additional m-$2N$ eigenvalues).

The right and left eigenproblems can be expressed, respectively, as follows:

$$(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{G}(\lambda_i) + \mathbf{K})\mathbf{u}_i = \mathbf{0} \quad \text{or} \quad \mathbf{D}(\lambda_i)\mathbf{u}_i = \mathbf{0} \tag{5}$$

$$(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{G}(\lambda_i) + \mathbf{K})^T\mathbf{v}_i = \mathbf{0} \quad \text{or} \quad \mathbf{D}(\lambda_i)^T\mathbf{v}_i = \mathbf{0} \tag{6}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ denotes, respectively, the right and left eigenvector corresponding to the ith eigenvalue λ_i. The system matrices are usually symmetric, but the possibility of the asymmetric matrices is also allowed in this note. For the self-adjoint case (symmetric matrices) the left and right eigenvectors are equal, $\mathbf{v}_i = \mathbf{u}_i$. Here we assume that the eigenvalues are distinct. To uniquely determine the eigenvectors, many approaches have proposed for normalizing the eigenvectors. Considering the consistent way with traditional modal analysis for undamped and viscous damped systems, the following normalization is adopted:

$$\mathbf{v}_i^T \frac{\partial \mathbf{D}(s)}{\partial s}\bigg|_{s=\lambda_i} \mathbf{u}_i = \theta_i \tag{7}$$

where $\theta_i \in \mathbb{C}$ is some nonzero constant. Equation (7) reduces to the corresponding normalization relationship for viscously damped systems when $\mathbf{G}(s)$ is constant with respect to s [6]. Numerical values of θ_i can be selected in several ways; the detailed discussions on the normalization of complex eigenvectors can be found in [3,12].

II. Eigensolution Derivatives: Self-Adjoint Case

In the self-adjoint case, the right and left eigenvectors are equal. The system matrices $\mathbf{M}$, $\mathbf{G}(\lambda_i)$ and $\mathbf{K}$ are assumed to depend continuously on the real design parameter p and their derivatives are known. Differentiating Eq. (5) with respect to the design parameter p, the eigenvector derivatives satisfy

$$\mathbf{D}(\lambda_i)\frac{\partial \mathbf{u}_i}{\partial p} + \boldsymbol{\alpha}_i \frac{\partial \lambda_i}{\partial p} = \mathbf{h}_i \tag{8}$$

where

$$\boldsymbol{\alpha}_i = \frac{\partial \mathbf{D}(s)}{\partial s}\bigg|_{s=\lambda_i}\mathbf{u}_i = \left(2\lambda_i\mathbf{M} + \mathbf{G}(\lambda_i) + \lambda_i\frac{\partial \mathbf{G}(s)}{\partial s}\bigg|_{s=\lambda_i}\right)\mathbf{u}_i$$

$$\mathbf{h}_i = -\left(\lambda_i^2\frac{\partial \mathbf{M}}{\partial p} + \lambda_i\frac{\partial \mathbf{G}(s)}{\partial p}\bigg|_{s=\lambda_i} + \frac{\partial \mathbf{K}}{\partial p}\right)\mathbf{u}_i$$

Equation (8) cannot be solved to find the eigenvector derivative because the matrix $\mathbf{D}(\lambda_i)$ is singular. For this reason, the normalization must be used. Differentiating Eq. (7) with $\mathbf{v}_i = \mathbf{u}_i$ yields

$$2\mathbf{u}_i^T\frac{\partial \mathbf{D}(s)}{\partial s}\bigg|_{s=\lambda_i}\frac{\partial \mathbf{u}_i}{\partial p} + \xi_i\frac{\partial \lambda_i}{\partial p} = c_i \tag{9}$$

where

$$\xi_i = \mathbf{u}_i^T\left(2\mathbf{M} + 2\frac{\partial \mathbf{G}(s)}{\partial s}\bigg|_{s=\lambda_i} + \lambda_i\frac{\partial^2 \mathbf{G}(s)}{\partial s^2}\bigg|_{s=\lambda_i}\right)\mathbf{u}_i$$

$$c_i = -\mathbf{u}_i^T\left(2\lambda_i\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{G}(s)}{\partial p}\bigg|_{s=\lambda_i} + \lambda_i\frac{\partial^2 \mathbf{G}(s)}{\partial p\,\partial s}\bigg|_{s=\lambda_i}\right)\mathbf{u}_i$$

Equations (8) and (9) can be rewritten in a linear system of algebraic equations:

$$\begin{bmatrix} \mathbf{D}(\lambda_i) & \boldsymbol{\alpha}_i \\ \mathbf{u}_i^T\frac{\partial \mathbf{D}(s)}{\partial s}\Big|_{s=\lambda_i} & \frac{\xi_i}{2} \end{bmatrix}\begin{Bmatrix} \frac{\partial \mathbf{u}_i}{\partial p} \\ \frac{\partial \lambda_i}{\partial p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{h}_i \\ \frac{c_i}{2} \end{Bmatrix} \tag{10}$$

or

$$\mathbf{A}_i\mathbf{x}_i = \mathbf{b}_i \tag{11}$$

Then, the derivatives of eigensolutions can be computed directly by solving the above algebraic system because the coefficient matrix $\mathbf{A}_i$ has a full rank matrix, as demonstrated in the Appendix. When $\mathbf{G}(\lambda_i)$ is constant with respect to λ_i, Eq. (11) reduces to the case of a viscously damped system. In this special case, Eq. (11) is similar with that developed in [7] and identical with [9], hence Eq. (11) may be considered as an extension of the algebraic method [7,9] to nonviscous damped systems.

III. Eigensolution Derivatives: Nonself-Adjoint Case

Two problems arise in the nonself-adjoint case: the left and right eigenvector derivatives must be computed simultaneously, and a supplementary constraint must be imposed to the relative scaling of the left and right eigenvectors (respectively, their derivatives).

Similarly, differentiating Eq. (6), the left eigenvector derivatives satisfy

$$\boldsymbol{\beta}_i\frac{\partial \lambda_i}{\partial p} + \mathbf{D}(\lambda_i)^T\frac{\partial \mathbf{v}_i}{\partial p} = \mathbf{f}_i \tag{12}$$

where

$$\boldsymbol{\beta}_i = \left(2\lambda_i\mathbf{M} + \mathbf{G}(\lambda_i) + \lambda_i\frac{\partial \mathbf{G}(s)}{\partial s}\bigg|_{s=\lambda_i}\right)^T\mathbf{v}_i$$

$$\mathbf{f}_i = -\left(\lambda_i^2\frac{\partial \mathbf{M}}{\partial p} + \lambda_i\frac{\partial \mathbf{G}(s)}{\partial p}\bigg|_{s=\lambda_i} + \frac{\partial \mathbf{K}}{\partial p}\right)^T\mathbf{v}_i$$

Due to $\mathbf{v}_i \neq \mathbf{u}_i$, the derivative of normalization as in Eq. (9) can be rewritten as

$$\mathbf{v}_i^T\frac{\partial \mathbf{D}(s)}{\partial s}\bigg|_{s=\lambda_i}\frac{\partial \mathbf{u}_i}{\partial p} + \tilde{\xi}_i\frac{\partial \lambda_i}{\partial p} + \mathbf{u}_i^T\frac{\partial [\mathbf{D}(s)]^T}{\partial s}\bigg|_{s=\lambda_i}\frac{\partial \mathbf{v}_i}{\partial p} = \tilde{c}_i \tag{13}$$

where

$$\tilde{\xi}_i = \mathbf{v}_i^T\left(2\mathbf{M} + 2\frac{\partial \mathbf{G}(s)}{\partial s}\bigg|_{s=\lambda_i} + \lambda_i\frac{\partial^2 \mathbf{G}(s)}{\partial s^2}\bigg|_{s=\lambda_i}\right)\mathbf{u}_i$$

$$\tilde{c}_i = -\mathbf{v}_i^T\left(2\lambda_i\frac{\partial \mathbf{M}}{\partial p} + \frac{\partial \mathbf{G}(s)}{\partial p}\bigg|_{s=\lambda_i} + \lambda_i\frac{\partial^2 \mathbf{G}(s)}{\partial p\,\partial s}\bigg|_{s=\lambda_i}\right)\mathbf{u}_i$$

However, this above additional equation is not enough to uniquely determine the derivatives of the left and right eigenvectors. Therefore, a supplementary constraint should be imposed. It can be noted that the left and right eigenvectors normalized by Eq. (7) are not determined uniquely. This can be demonstrated by multiplying the left eigenvector by any scalar and dividing the right eigenvector by the same scalar; Eq. (7) is also satisfied. The best approach is to set one component in both eigenvectors to be equal. This component is arbitrary, but should be chosen such that this component has a large magnitude in both the left and right eigenvectors. One possibility is to multiply the magnitudes of the corresponding components of both eigenvectors and choose the largest product. Suppose that vector component number n_i is chosen, thus

$$\{\mathbf{u}_i\}_{n_i} = \{\mathbf{v}_i\}_{n_i}, \qquad \left\{\frac{\partial \mathbf{u}_i}{\partial p}\right\}_{n_i} = \left\{\frac{\partial \mathbf{v}_i}{\partial p}\right\}_{n_i} \tag{14}$$

The above supplementary constraint can be rewritten by

$$\mathbf{W}_i\frac{\partial \mathbf{u}_i}{\partial p} = \mathbf{W}_i\frac{\partial \mathbf{v}_i}{\partial p} \tag{15}$$

where

$$\mathbf{W}_i = \underbrace{\left\{0 \quad \cdots \quad 0 \quad \overbrace{\kappa}^{n_i\text{th column}} \quad 0 \quad \cdots \quad 0\right\}}_{N}$$

$\mathbf{W}_i$ is a $(1 \times N)$ weight vector and the n_ith component of it associated with the ith left or right eigenvector is set to a nonzero constant κ. To reduce the condition number, select the nonzero constant κ by finding the absolutely largest element of matrix $\mathbf{D}(\lambda_i)$.

Rewriting Eqs. (8), (12), (13), and (15) in the following matrix form:

$$\begin{pmatrix} \mathbf{D}(\lambda_i) & \boldsymbol{\alpha}_i & \mathbf{W}_i^T & \mathbf{0} \\ \mathbf{v}_i^T\frac{\partial \mathbf{D}(s)}{\partial s}\Big|_{s=\lambda_i} & \tilde{\xi}_i & 0 & \mathbf{u}_i^T\frac{\partial [\mathbf{D}(s)]^T}{\partial s}\Big|_{s=\lambda_i} \\ \mathbf{W}_i & 0 & 0 & -\mathbf{W}_i \\ \mathbf{0} & \boldsymbol{\beta}_i & -\mathbf{W}_i^T & \mathbf{D}(\lambda_i)^T \end{pmatrix}\begin{Bmatrix} \frac{\partial \mathbf{u}_i}{\partial p} \\ \frac{\partial \lambda_i}{\partial p} \\ 0 \\ \frac{\partial \mathbf{v}_i}{\partial p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{h}_i \\ \tilde{c}_i \\ 0 \\ \mathbf{f}_i \end{Bmatrix} \tag{16}$$

or

$$\tilde{\mathbf{A}}_i\tilde{\mathbf{x}}_i = \tilde{\mathbf{b}}_i \tag{17}$$

The eigensolution derivatives can be computed directly by solving the above algebraic system because the coefficient matrix $\tilde{\mathbf{A}}_i$ is a full rank matrix (see the Appendix). As it can be noted, only one equation is needed to compute the eigensolution derivatives, therefore, the method is very simple and compact. Naturally, in the case of symmetric damped systems, the left and right eigenvectors are identical, and the new supplementary constraint expressed in Eq. (15) will be ignored because it is always satisfied. In this special case, Eq. (16) will reduce to Eq. (11) (the case of symmetric nonviscously damped systems).

IV. Numerical Example

To illustrate the validity of the approach proposed, a 2 DOF nonviscously damped system [3,6] is considered. The equations of

Table 1 Eigensolutions and their derivatives with respect to the stiffness parameter k_1

Quantity	Elastic mode 1	Elastic mode 2	Nonviscous mode 1	Nonviscous mode 2
λ_i	$-0.0387 \pm 38.3232i$	$-1.5450 \pm 97.5639i$	-2.8403	-5.9923
$\mathbf{u}_i$	$\begin{Bmatrix} -0.7500 \pm 0.0043i \\ -0.6616 \mp 0.0041i \end{Bmatrix}$	$\begin{Bmatrix} 0.6622 \mp 0.0035i \\ -0.7501 \pm 0.0075i \end{Bmatrix}$	$\begin{Bmatrix} -0.0165 \\ 0.0083 \end{Bmatrix}$	$\begin{Bmatrix} 0.0055 \\ -0.0028 \end{Bmatrix}$
$\partial\lambda_i/\partial k_1$	$-9.1317 \times 10^{-5} \pm 7.3385 \times 10^{-3}i$	$5.9174 \times 10^{-5} \pm 2.2461 \times 10^{-3}i$	-2.7088×10^{-4}	-2.9832×10^{-5}
$\partial\mathbf{u}_i/\partial k_1$	$\begin{Bmatrix} 1.1304 \times 10^{-4} \mp 6.6079 \times 10^{-6}i \\ 1.6934 \times 10^{-5} \pm 4.1016 \times 10^{-6}i \end{Bmatrix}$	$\begin{Bmatrix} 3.8526 \times 10^{-5} \mp 1.5156 \times 10^{-6}i \\ 4.9412 \times 10^{-5} \mp 2.5627 \times 10^{-6}i \end{Bmatrix}$	$\begin{Bmatrix} 7.2075 \times 10^{-6} \\ 4.6049 \times 10^{-6} \end{Bmatrix}$	$\begin{Bmatrix} -1.8408 \times 10^{-6} \\ -1.7653 \times 10^{-6} \end{Bmatrix}$

motion describing the free vibration of the system can be expressed by Eq. (2), with

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_3 + k_2 \end{bmatrix},$$

$$\mathbf{G}(s) = c\{(1 + s/\mu_1)^{-1} + (1 + s/\mu_2)^{-1}\}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where $m = 1$ kg, $k_1 = 1000$ N/m, $k_2 = 2000$ N/m, $k_3 = 1600$ N/m, $c = 200$ Ns/m, $\mu_1 = 5$ s^{-1} and $\mu_1 = 7$ s^{-1}.

The normalization constants θ_i are selected such that $\theta_i = 2\lambda_i$ are for the elastic modes and $\theta_i = 1$ are for the nonviscous modes. Adhikari's method [11] was used to compute the eigenvalues and eigenvectors. The derivatives of eigensolutions are uniquely determined from Eq. (11) and shown in Table 1. As it can be noted, the results are exactly the same as that in [6].

V. Conclusions

A method has been outlined to compute simultaneously and accurately the derivatives of eigenvalues and their associated eigenvectors for the systems with nonviscous damping. The algebraic method used, which requires only the eigenvector of interest, gives exact eigensolution derivatives. Moreover, the proposed method is easy to implement because it is very compact. A numerical example has demonstrated the validity of the proposed approach.

Appendix A: Numerical Stability

In this Appendix, the coefficient matrix $\mathbf{A}_i$ will be proved to be a full rank matrix. Assume $\mathbf{A}_i\mathbf{s} = 0$ for $\mathbf{s} = \begin{bmatrix} \Theta \\ \tau \end{bmatrix} \in \mathbb{C}^{(N+1)\times 1}$. That is

$$\mathbf{D}(\lambda_i)\Theta + \boldsymbol{\alpha}_i\tau = 0 \tag{A1}$$

$$2\mathbf{u}_i^T \left.\frac{\partial \mathbf{D}(s)}{\partial s}\right|_{s=\lambda_i} \Theta + \xi_i\tau = 0 \tag{A2}$$

Premultiplying Eq. (A1) by $\mathbf{u}_i^T$ and using Eq. (5) yields

$$\mathbf{u}_i^T \left.\frac{\partial \mathbf{D}(s)}{\partial s}\right|_{s=\lambda_i} \mathbf{u}_i\tau = 0 \tag{A3}$$

One can obtain τ is equal to a zero from Eq. (7). Substituting $\tau = 0$ into Eq. (A1), there is

$$\mathbf{D}(\lambda_i)\Theta = \mathbf{0} \tag{A4}$$

Obviously, $\mathbf{u}_i$ is a particular solution of the above equation. Thus, Θ can be expressed by

$$\Theta = c_i\mathbf{u}_i \tag{A5}$$

where c_i are constant coefficients. Substituting Eq. (A5) into Eq. (A2), and utilizing normalization Eq. (7), one can obtain $c_i = 0$, thus, $\mathbf{s} = \mathbf{0}$. The equation $\mathbf{A}_i\mathbf{s} = \mathbf{0}$ has the unique solution $\mathbf{s} = \mathbf{0}$; hence, it can be concluded that $\mathbf{A}_i$ is always a full rank matrix. The coefficient matrix $\tilde{\mathbf{A}}_i$ of Eq. (17) can also be proved to be a full rank matrix; the proof will not be given herein, because it can be proved based on the similar procedures above.

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